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THE SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AS FUNCTIONS OF THEIR INITIAL VALUES

BY G. A. BLISS

THE proof of the existence of solutions of the differential equation

$$(1) \qquad \frac{dx}{dt} = f(t, x),$$

where the function f is expansible into a power-series in t and x, is included in most of the text books on differential equations. Methods have also been devised for proving the existence of an integral taking a given value ξ for $t = \tau$, when the function f satisfies much less stringent restrictions.* In many applications however the mere existence theorem is of little value unless accompanied by a proof of the continuity and differentiability of the solution considered as a function of the initial constants τ, ξ. As the literature dealing with these latter questions is somewhat scattered and the proofs complicated, it is proposed in the following pages to collect and simplify as far as possible the results of the different writers. Solutions are defined reaching from boundary to boundary of the region R over which the properties of the function f are assumed provided R is closed, and the method of Peano† for proving the existence and continuity of the first partial derivatives with respect to the initial constants has been extended by a simple device to the higher derivatives.

1. The existence of solutions. In order to make the existence proof suppose 1) that the function $f(t, x)$ in equation (1) is continuous in the interior of a region‡ R of the tx-plane, and 2) that a positive constant k' can be found for any finite closed region R' interior to R, such that

$$(2) \qquad |f(t, x) - f(t, x')| \leqq k'|x - x'|,$$

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* See for example, Osgood, *Monatshefte für Mathematik*, vol. 9, 1898. Only the continuity of $f(t, x)$ is assumed. For other references, see *Encyclopädie der mathematischen Wissenschaften*, IIA 4a, especially pp. 195, 200.

† *Atti di Torino*, vol. 33, p. 9 ff.

‡ In the following pages the word region means always a point set with interior points. A closed region is one which contains all its limit points.

where (t, x) and (t, x') are any two points for which the interval $[x, x']$ on the t-ordinate consists only of points in R'.*

Consider the series of approximation functions :†

$$\left\{\begin{array}{l} x^{(1)} = \xi + \int_\tau^t f(t, \xi)\,dt \\ x^{(2)} = \xi + \int_\tau^t f(t, x^{(1)})\,dt \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x^{(n)} = \xi + \int_\tau^t f(t, x^{(n-1)})\,dt \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \end{array}\right. \tag{3}$$

where (τ, ξ) is a point interior to the region R through which it is desired to pass a solution of equation (1). Any one $x^{(n)}$ of these functions is surely well-defined on an interval

$$|t - \tau| \leqq l \tag{4}$$

for which the preceding function $x^{(n-1)}$ is continuous and the points $(t, x^{(n-1)})$ all lie in the region R, and each function takes the initial value ξ when $t = \tau$. An interval (4) can be found by selecting a positive constant ρ so that all the points which satisfy the inequalities

$$R_1: \qquad |t - \tau| \leqq \rho, \quad |x - \xi| \leqq \rho \tag{5}$$

are interior to the region R, and then taking l as the smaller of ρ and $\frac{\rho}{M_1}$, where M_1 is the maximum of the absolute value of f in the neighborhood (5) of the point (τ, ξ). For then it follows from the relations

$$|x^{(n)} - \xi| = \left|\int_\tau^t f(t, x^{(n-1)})\,dt\right| \leqq M_1 |t - \tau| \leqq \rho, \qquad (n = 1, 2, \cdots),$$

that all the points $(t, x^{(1)})$, $(t, x^{(2)})$, etc., are in the neighborhood (5).

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* This is the so-called Lipschitz condition (see *Encyclopädie der Math. Wiss.*, IIA 4a, p. 194). It will certainly be satisfied if f has a continuous partial derivative $f_x = \frac{\partial f}{\partial x}$ in R. For then

$$f(t, x) - f(t, x') = (x - x')\, f_x(t, x + \theta(x - x')), \qquad 0 < \theta < 1$$

provided that the interval $[x, x']$ of the t-ordinate is all in R', and the maximum of $|f_x|$ in R' would be a constant k of the kind specified.

† Compare Picard, *Traité d'analyse*, vol. 2, p. 301.

The series

$$x = \xi + (x^{(1)} - \xi) + (x^{(2)} - x^{(1)}) + \cdots \tag{6}$$

is uniformly convergent* for all values of t on the interval (4). Namely from (2),

$$\begin{aligned} |x^{(n)} - x^{(n-1)}| &\leqq \left| \int_\tau^t \{ f(t, x^{(n-1)}) - f(t, x^{(n-2)}) \} dt \right| \\ &\leqq \left| \int_\tau^t k_1 | x^{(n-1)} - x^{(n-2)} | dt \right|, \qquad (n > 1), \end{aligned} \tag{7}$$

where k_1 is the constant in (2) belonging to the neighborhood R_1. Let N be the maximum of $|f(t, \xi)|$ on the interval (4). Then

$$|x^{(1)} - \xi| \leqq N |t - \tau|,$$

and by successive application of the inequality (7),

$$|x^{(n)} - x^{(n-1)}| \leqq \frac{N}{k_1} \frac{k_1^n |t - \tau|^n}{n!}.$$

The absolute value of any term of the series (6) is therefore less than the corresponding term of the development of $\frac{N}{k_1} \{ e^{k_1 |t - \tau|} - 1 \}$, which is a series of positive terms converging uniformly for all values of t in the interval (4). From this the uniform convergence of (6) follows.

The continuous function $x(t)$ defined by the series (6) *on the interval* (4), *takes the value ξ for $t = \tau$ and satisfies the differential equation* (1). For since the region (5) is a closed one the function f is uniformly continuous,† *i. e.* for any positive ϵ a positive δ can be found such that

$$|f(t', x') - f(t, x)| < \epsilon,$$

where (t', x') and (t, x) are any two points of R_1 satisfying

$$|t' - t| < \delta, \quad |x' - x| < \delta.$$

Furthermore on account of the uniform convergence of the series (6) a positive integer m can be found such that if $n > m$,

$$|x^{(n-1)} - x| < \delta.$$

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* Here, as in what follows, the fundamental theorems concerning uniform convergence are assumed. See, *e. g.*, Jordan, *Cours d'analyse*, vol. 1, p. 310 ff.

†A function which is continuous in a finite closed region, is uniformly continuous; see Jordan, *loc. cit.*, p. 48.

Therefore also

$$\left| \int_{\tau}^{t} \{ f(t, x^{(n-1)}) - f(t, x) \} dt \right| < \epsilon l$$

for $n > m$, and

$$\lim_{n = \infty} \int_{\tau}^{t} f(t, x^{(n-1)}) dt = \int_{\tau}^{t} f(t, x) dt.$$

By letting n approach infinity on both sides of the equation (3), it follows that

$$x = \xi + \int_{\tau}^{t} f(t, x) dt, \tag{8}$$

from which the statements made above are easily proved by putting $t = \tau$ and by differentiation.

A solution having been defined in the interval $[\tau, \tau + l]$, the same process can be applied again with the point $\big(\tau + l,\ x(\tau + l)\big)$ as initial point, provided that this point is still within R, and the solution can be extended to an interval $[\tau + l,\ \tau + l + l']$. If the region R is finite and closed the values of t which can be reached by such continuations must have an upper bound t_1, and the points $\big(t,\ x(t)\big)$ approach a definite limit point as t approaches t_1. For if t' and t'' both satisfy the inequality

$$0 < t_1 - t < \frac{\epsilon}{M},$$

where M is the maximum of $|f|$ in the region R, then equation (8) shows that

$$|x(t') - x(t'')| \leqq M\, |t' - t''| < \epsilon, \tag{9}$$

and this is the necessary and sufficient condition that the function $x(t)$ has a unique limiting value for $t = t_1$.* The limit point $\big(t_1,\ x(t_1)\big)$ must be a boundary point of the region R. Otherwise it would be an interior point, and in that case t_1 would not be the upper bound of the t-values which could be reached by continuation.

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*Suppose $t', t'', \ldots, t^{(n)}, \ldots$ have the limit t_1 and are less than t_1. By Jordan, *loc. cit.*, p 9, the corresponding values of x define a unique limit. But by (9) this is the limit which x approaches as t approaches t_1 over any set of values whatsoever.

On the other hand, if the region R is not finite and closed, it might happen that the solution could be extended over any t-interval, or if t_1 were finite perhaps the function $x(t)$ would have no unique limit at $t = t_1$. In this case also, however, if t_1 is finite, and if $x(t)$ approaches a unique limit as t approaches t_1, then it can be proved as above that the point $\big(t_1,\ x(t_1)\big)$ must be a boundary point of R.

The preceding discussion justifies then the following theorem:

Through any point (τ,ξ) *in the interior of the region* R *in which the function* f *is continuous and satisfies the Lipschitz condition* (2), *a solution of equation* (1), *denoted by*

$$x = \phi(t,\tau,\xi), \tag{10}$$

can be passed. The function ϕ *is continuous in* t *and defines points* (t,x) *in the interior of* R *for all values of* t *in an interval*

$$t_0 < t < t_1,$$

in which the value τ *is included.*

If the region R *is finite and closed, the interval* $[t_0,t_1]$ *is finite and can be chosen so that as* t *approaches* t_0 (*or* t_1), *the points* (t,x) *of the solution approach a unique limit point on the boundary of the region.*

If R *is not finite and closed, the largest possible interval* $[t_0,\ t_1]$ *may be infinite, or if finite the solution may not have definite limit points at its extremities* t_0 *and* t_1. *In case* t_0 (*or* t_1) *is finite and the solution approaches a unique limit point, then as before the limit point must belong to the boundary of* R.

For some applications it is important to notice that *if* P *is a finite closed region entirely interior to* R, *a positive quantity* λ *can be determined, such that the solution through any point* $(\tau,\ \xi)$ *of* P *can be extended at least over the interval* $|t-\tau| \leqq \lambda$. This follows because a constant ρ can be chosen independent of the position of $(\tau,\ \xi)$ in P and so that the corresponding region (5) is entirely interior to R. If M is the maximum of $|f|$ in the region consisting of all the regions (5) about points of P, then according to the preceding discussion the smaller of ρ and $\frac{\rho}{M}$ will be a constant λ with the desired property.

2. Uniqueness of the solution.* Consider a solution $S\big(x = x(t)\big)$ of equation (1), defined and interior to the region R for all values of t in an interval $[t', t'']$. A positive constant ρ can be so selected that all points of the neighborhood R_2 of S indicated by the dotted lines in the figure, are also interior to R. Let $\bar{S}\Big(y = y(t)\Big)$ be another solution such that the value $t = \tau$,

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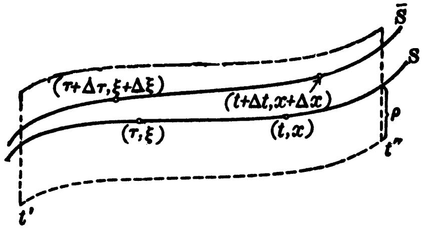
*Compare the methods used in §§2, 3, 4 with Peano, *loc. cit.*



and therefore all values in a certain interval T including τ, define points (t, y) interior to the region R. Then the difference $(x - y)$ satisfies the equation

$$\frac{d(x-y)}{dt} = f(t, x) - f(t, y),$$

and the forward derivative* of $|x - y|$ satisfies the inequality

$$\left|\frac{d|x-y|}{dt}\right| = \left|\frac{d(x-y)}{dt}\right| \leqq k_2|x-y| \tag{11}$$

on account of the condition (2), where k_2 is the value of k belonging to the region R_2. Consequently

$$\frac{d|x-y|e^{-k_2(t-\tau)}}{dt} \leqq 0, \quad \frac{d|x-y|e^{k_2(t-\tau)}}{dt} \geqq 0.$$

By using the former inequality for $t > \tau$ and the latter for $t < \tau$, it follows that

$$|x-y| \leqq |\xi - \eta| e^{k_2|t-\tau|} \tag{12}$$

for all values of t on the interval T, where ξ and η are the values of x and y

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*The forward derivative is used because at points where a function $u(t)$ vanishes, the derivative of $|u| = +\sqrt{u^2}$ is discontinuous. At such values of t, however, the *forward* derivative exists, and by the mean-value theorem is

$$\frac{d\sqrt{u^2}}{dt} = \lim_{h=0} \frac{1}{h}\sqrt{[hu'(t+\theta h)]^2} = +\sqrt{\left(\frac{du}{dt}\right)^2},$$

where h is supposed to approach zero from the positive side. At other points the first part of (11) is easily proved.

for $t = \tau$.* It follows, then, that if S and $\bar{S}$ intersect at any point $t = \tau$, $\xi = \eta$, the value of $|x - y|$ is less than or equal to zero, and x must be identically equal to y for all values of t in the common interval adjoining τ in which the two solutions are defined.

Only one solution of equation (1) *can pass through a given point* (τ, ξ) *interior to* R. *The set* (10) *includes, therefore, all the solutions in the region.*

3. Continuity of the solutions with respect to the initial values. With the help of the inequality (12) it can be shown that the solution (10) through the point (τ, ξ) can be surrounded by solutions corresponding to neighboring initial values $(\tau + \Delta\tau, \xi + \Delta\xi)$, and that these go over continuously into the former when $\Delta\tau$ and $\Delta\xi$ approach zero. Let S denote the solution (10), and suppose that an interval $[t', t'']$ and a neighborhood R_2 of the kind described in §2 have been found. $\Delta\tau$ and $\Delta\xi$ are to be so chosen that the point $(\tau + \Delta\tau, \xi + \Delta\xi)$ lies in R_2, and $\bar{S}$ will be used to denote the solution

$$x = \phi(t, \tau + \Delta\tau, \xi + \Delta\xi)$$

through that point. Then from equation (8)

$$\begin{aligned} |\phi(\tau + \Delta\tau, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(\tau + \Delta\tau, \tau, \xi)| &= \left|\Delta\xi - \int_{\tau}^{\tau+\Delta\tau} f(t, \phi)\, dt\right| \\ &\leqq |\Delta\xi| + M_2|\Delta\tau|, \end{aligned}$$

where M_2 is the maximum of $|f|$ in the neighborhood R_2. This gives the difference of the values of x for S and $\bar{S}$ at $t = \tau + \Delta\tau$, so that when τ is replaced by $\tau + \Delta\tau$, the inequality (12) becomes

$$(13)\qquad |\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)| \leqq \{|\Delta\xi| + M_2|\Delta\tau|\} e^{k_2|t-\tau-\Delta\tau|}.$$

Suppose now that $\Delta\tau$ and $\Delta\xi$ are still further restricted so that

$$(14)\qquad \{|\Delta\xi| + M_2|\Delta\tau|\} e^{k_2|t''-t'|} < \rho.$$

Then the solution $\bar{S}$ through $(\tau + \Delta\tau, \xi + \Delta\xi)$ can be continued over the whole

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* *If a function $f(t)$ is continuous and has a forward derivative $f'(t) \leqq 0$ on the interval from t_0 to t_1 $(t_0 < t_1)$, then $f(t_0) \geqq f(t_1)$.* For, the function

$$\phi(t) = f(t) + \beta t,$$

where β is any negative constant, has a negative forward derivative. If $\phi(t_0)$ were less than $\phi(t_1)$, the function ϕ would attain its minimum at some value t' less than t_1. At that point the forward derivative would necessarily be $\geqq 0$, contrary to the hypothesis. Consequently

$$\phi(t_0) - \phi(t_1) = f(t_0) - f(t_1) - \beta(t_1 - t_0) > 0$$

for every negative β, which proves the theorem stated for $f(t)$.

interval $t't''$. For by the results of §1, the process of continuation can be carried on until the boundary of R_2 is reached, and on account of the inequalities (13) and (14) the only points attained by such continuations between the ordinates $t = t'$ and $t = t''$ are in the interior of R_2. The ordinates $t = t'$ and $t = t''$ are therefore the only boundaries which the solution $\bar{S}$ can cross.

From these results it follows that if t, $t + \Delta t$ are any two values in the interval $[t', t'']$, then by applying (8) and (13),

$$\begin{aligned} |\Delta\phi| &= |\phi(t + \Delta t, \ \tau + \Delta\tau, \ \xi + \Delta\xi) - \phi(t, \tau, \xi)| \\ &\leqq |\phi(t + \Delta t, \ \tau + \Delta\tau, \ \xi + \Delta\xi) - \phi(t, \tau + \Delta\tau, \ \xi + \Delta\xi)| \\ &\qquad + |\phi(t, \tau + \Delta\tau, \ \xi + \Delta\xi) - \phi(t, \tau, \xi)| \\ &\leqq M_2|\Delta t| + \{ |\Delta\xi| + M_2|\Delta\tau| \} e^{k_2(t-\tau-\Delta\tau)}. \end{aligned}$$

Therefore the function (10) *is continuous for all values of* t, τ, ξ *defining points of the solution interior to the region* R. For, as has been indicated, an interval $[t', t'']$ including the values t and τ can always be selected, and $\Delta\tau$, $\Delta\xi$ restricted, so that the solutions $\bar{S}$ are defined on the interval and satisfy the last inequality.

4. Partial derivatives with respect to the constants of integration. In order to prove the existence of the partial derivatives $\frac{\partial\phi}{\partial\tau}$, $\frac{\partial\phi}{\partial\xi}$ of the function (10) at a point t, τ, ξ, the additional assumption is made that the derivative $f_x = \frac{\partial f}{\partial x}$ exists and is continuous in the interior of the region R. For the two solutions S, $\bar{S}$ of §3, it follows from equation (1) that

$$\frac{d\Delta\phi}{dt} = a\Delta\phi, \tag{15}$$

where

$$\Delta\phi = \phi(t, \ \tau + \Delta\tau, \ \xi + \Delta\xi) - \phi(t, \tau, \xi),$$

$$a = \frac{f(t, \phi + \Delta\phi) - f(t, \phi)}{\Delta\phi}.$$

In these expressions τ and ξ are considered fixed, for the moment, while t, $\Delta\tau$, $\Delta\xi$ are variable. The coefficient a is a continuous function of t, $\Delta\tau$, $\Delta\xi$ when t is on the interval $[t', t'']$, and $\Delta\tau$, $\Delta\xi$ satisfy (14).* This follows from §3 for any values of the arguments for which $\Delta\phi \neq 0$, and with the help of the expression

$$a = f_x(t, \phi + \theta\Delta\phi), \qquad 0 < \theta < 1,$$

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*See a note by Hadamard, *Bull. de la société math.*, vol. 28 (1900), p. 64.

for any values for which $\Delta\phi = 0$, on account of the continuity of the derivative f_x. The general integral of equation (15) is therefore

$$\Delta\phi = ce^{\int_\tau^t a\,dt}$$

When $\Delta\tau = 0$, the constant c has the value $\Delta\xi$, and it follows easily that the quotient $\frac{\Delta\phi}{\Delta\xi}$ has the limit

$$\frac{\partial\phi}{\partial\xi} = e^{\int_\tau^t f_x(t,\,\phi)dt}\,, \tag{16}$$ *

which is a continuous function of t, τ, ξ on account of the continuity of ϕ and f_x. Similarly when $\Delta\xi = 0$,

$$\begin{aligned} c = \Delta\phi\,|^{t=\tau} &= \int_{\tau+\Delta\tau}^{\tau} f(t,\,\phi + \Delta\phi)\,dt \\ &= -\,\Delta\tau f\Big(\tau + \theta\Delta\tau,\ \phi(\tau + \theta\Delta\tau,\,\tau + \Delta\tau,\ \xi)\Big),\ 0 < \theta < 1, \end{aligned}$$

by (8) and the mean value theorem for a definite integral. At the limit therefore

$$\frac{\partial\phi}{\partial\tau} = -f(\tau,\ \xi)e^{\int_\tau^t f_x(t,\,\phi)dt} \tag{17}$$

Differentiation of equation (8) twice for t shows that it the function f has the derivative $\frac{\partial f}{\partial t}$ continuous in the interior of R, then the derivative $\frac{\partial^2\phi}{\partial t^2}$ exists and is also continuous. If furthermore all the second derivatives of f except perhaps $\frac{\partial^2 f}{\partial t^2}$ are continuous, the equations (16) and (17) can be differentiated, and ϕ will have all the derivatives of the second order. A simple induction leads to the theorem:

When the function $f(t, x)$ has all its derivatives of the $(n-1)^{\text{st}}$ order, and all of the n^{th} order except perhaps $\frac{\partial^n f}{\partial t^n}$, continuous in the interior of the region R, then the solution (10) *has continuous derivatives of the n^{th} order with respect to t, τ, ξ at any set of values (t, τ, ξ) defining a point (t, x) of the solution interior to R.*

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* See Bendixon, *Bull. de la société math.*, vol. 24 (1896), p. 220.

5. Complex numbers.* The results of the preceding sections can be readily extended to a *system* of differential equations by means of complex numbers and matrices. The necessary definitions follow:

1) x is a symbol for the set $(x_1, x_2, \cdots, x_r)$;

2) $\operatorname{mod} x = \sqrt{x_1^2 + x_2^2 + \cdots + x_r^2}$;

3) $x \gtreqless y$ if $x_i \gtreqless y_i \quad (i = 1, 2, \cdots r)$;

4) $x \pm y$ are the complex numbers $(x_1 \pm y_1, x_2 \pm y_2, \cdots, x_r \pm y_r)$;

5) xy is the sum $\sum\limits_{i=1}^{r} x_i y_i$;

6) kx is the complex number $(kx_1, kx_2, \cdots, kx_r)$ if k is a simple constant;

7) if the elements of x are functions of t, then $\int x dt$ and $\frac{dx}{dt}$ are the complex numbers $\left(\int x_1 dt, \int x_2 dt, \cdots, \int x_r dt\right)$ and $\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_r}{dt}\right)$ respectively;

8) $|x|$ is the complex number $(|x_1|, |x_2|, \cdots, |x_r|)$;

9) A denotes a matrix of elements $a_{ij} \quad (i, j = 1, 2, \cdots, r)$;

10) Ax is the complex number of which the elements are $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ir}x_r \quad (i = 1, 2, \cdots, r)$;

11) $A = B$ if $a_{ij} = b_{ij} \quad (i, j = 1, 2, \cdots, r)$;

12) AB is the matrix C whose elements are $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} \quad (i, j = 1, 2, \cdots, r)$;

13) $\frac{dA}{dt}$ is the matrix whose elements are $\frac{da_{ij}}{dt} \quad (i, j = 1, 2, \cdots r)$;

14) $\frac{dx}{d\xi}$, when x and ξ are two complex numbers such that the elements of x are functions of the elements of ξ, is the matrix of derivatives $\frac{\partial x_i}{\partial \xi_j} \quad (i, j = 1, 2, \cdots r)$.

Besides these definitions the following three theorems are needed:

a) $\operatorname{mod}(x \pm y) \leqq \operatorname{mod} x + \operatorname{mod} y$;

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*For the application of complex numbers to linear equations, see Peano, *Math. Annalen*, vol. 32 (1888), p. 450. In the article by Peano cited previously, they are used without explanation for any system of equations in the so-called normal form.

b) if the elements of x are functions of t having continuous derivatives, then mod x has a forward derivative and

$$\left|\frac{d \operatorname{mod} x}{dt}\right| \leqq \operatorname{mod}\frac{dx}{dt} ;$$

c) $\operatorname{mod}\int_\tau^t x dt \leqq \left|\int_\tau^t \operatorname{mod} x dt\right|.$

To prove *a*) it is only necessary to apply the identity

$$(18) \qquad \sum_{i=1}^{r} x_i^2 \cdot \sum_{i=1}^{r} y_i^2 = \left(\sum_{i=1}^{r} x_i y_i\right)^2 + \sum_{i,j=1}^{r} (x_i y_j - x_j y_i)^2,$$

after squaring both sides of the inequality. To prove *b*) two kinds of t-points must be considered, according as $x \neq 0$ or $x = 0$. For the former [see 5)]

$$\left|\frac{d \operatorname{mod} x}{dt}\right| = \left|\frac{x\dfrac{dx}{dt}}{\operatorname{mod} x}\right| \leqq \operatorname{mod}\frac{dx}{dt},$$

on account of (18) when $y = \dfrac{dx}{dt}$; and for the latter the forward derivative by the mean value theorem is

$$\frac{d \operatorname{mod} x}{dt} = \lim_{h=0} \frac{1}{h} \sqrt{\sum_{i=1}^{r} [h x_i'(t + \theta h)]^2} = + \sqrt{\sum_{i=1}^{r} \left(\frac{dx}{dt}\right)^2},$$

(see footnote p. 54) where h approaches zero from the positive side. From *b*) it follows further that

$$\left|\frac{d}{dt} \operatorname{mod} \int_\tau^t x dt\right| \leqq \operatorname{mod} x = \frac{d}{dt}\int_\tau^t \operatorname{mod} x dt.$$

Therefore

$$\frac{d}{dt}\left\{\operatorname{mod}\int_\tau^t x dt - \int_\tau^t \operatorname{mod} x dt\right\} \leqq 0, \quad \frac{d}{dt}\left\{\operatorname{mod}\int_\tau^t x dt + \int_\tau^t \operatorname{mod} x dt\right\} \geqq 0,$$

and by using the former for $t > \tau$ and the latter for $t < \tau$, it is seen that *c*) must be true on account of the footnote on p. 55.

6. Systems of differential equations. By means of the definitions of §5 a system of differential equations

$$(19) \qquad \frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots, x_r), \qquad i = 1, 2, \cdots, r),$$

can be represented by a single equation of the form (1), where however x and f are complex with the elements x_i and f_i respectively. It is presupposed that the functions f_i are continuous in the interior of a region R of $r+1$ dimensions, and that a positive constant k' exists for any finite closed region R' interior to R, such that

$$(20) \qquad \operatorname{mod}[f(t, x) - f(t, y)] \leqq k' \operatorname{mod}(x - y)$$

whenever (t, x), (t, y), and all the points (t, z) for which the elements z_i are in the interval $[x_i, y_i]$, lie in the region R'.*

If ξ is a complex number and (τ, ξ) is any point interior to R, a positive constant ρ can always be found such that all points (t, x) satisfying

$$R_1: \qquad |t - \tau| \leqq \rho, \quad \operatorname{mod}(x - \xi) \leqq \rho,$$

are also interior points. The set of approximation functions for the system (19) is defined by equations of the same form as equations (3) but with $x^{(n-1)}$, $x^{(n)}$, ξ, and f complex. If t remains on the interval

$$(21) \qquad |t - \tau| \leqq l,†$$

where l is the smaller of ρ and $\frac{\rho}{M_1}$, M_1 being the maximum of mod f‡ in the

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*The analogue to the Lipschitz condition of §1 would be, in the notation of complex numbers [see §4, 8), 10), and 3)],

$$(21) \qquad |f(t, x) - f(t, y)| \leqq K\, |x - y|,$$

where K is a matrix of positive constants k_{ij}. The condition given in the text is a consequence of this, as can be shown by taking the moduli of each side of the equation (21) and applying the relation

$$\left| \sum_{i,j=1}^{r} a_{ij} x_i x_j \right| \leqq \sum_{i,j=1}^{r} |a_{ij}| \sum_{i=1}^{r} x_i^2,$$

which holds for any quadratic form because $\left| x_i \Big/ \sqrt{\sum_{j=1}^{r} x_j^2} \right| \leqq 1$.

In case the elements f_i have continuous partial derivatives with respect to the variables x, the value of k_{ij} can be taken equal to the maximum of the absolute value of $f_{ij} = \frac{\partial f_i}{\partial x_j}$ $(i, j = 1, 2, \ldots, r)$ in the region R. For

$$f_i(t, x_1, \ldots, x_r) - f_i(t, y_1, \ldots, y_r) = \sum_{j=1}^{r} (x_j - y_j) f_{ij}\big(t, y_1 + \theta_1(x_1 - y_1), \ldots, x_r + \theta_r(x_r - y_r)\big),$$

where the θ's are all between zero and unity.

†It should be noticed that if the system of equations is linear in the x_i's this restriction on t is unnecessary. The approximation functions are well-defined on any interval for which the coefficients of the x_i's are continuous functions of t. (See §6.)

‡In many cases the proofs for a system of equations can be worded as for a single equation provided that absolute values are replaced by moduli as defined in §4.

region R_1, then by means of §5 c) it follows step by step that the functions $x^{(n)}$ satisfy the relations

$$\operatorname{mod}(x^{(n)} - \xi) \leqq M_1 |t - \tau| \leqq \rho,$$

and are well-defined and continuous in the interval (21).

In a manner analogous to that of §1, it can be shown with the help of (20) and §5 c) that the terms of the series

$$(22) \qquad \operatorname{mod}(x^{(1)} - \xi) + \operatorname{mod}(x^{(2)} - x^{(1)}) + \operatorname{mod}(x^{(3)} - x^{(2)}) + \cdots$$

are less than the corresponding terms of

$$\frac{N}{k_1}\left\{k_1|t-\tau| + \frac{k_1^2|t-\tau|^2}{2\,!} + \frac{k_1^3|t-\tau|^3}{3\,!} + \cdots\right\},$$

where N is the maximum of mod $f(t, \xi)$ in the interval (21), and the series (22) converges therefore uniformly on the interval (21). The r series defined by

$$(23) \qquad x = \xi + (x^{(1)} - \xi) + (x^{(2)} - x^{(1)}) + \cdots$$

are all convergent, and as in §1,

$$(24) \qquad x = \xi + \int_\tau^t f(t, x)\,dt.$$

Hence *the series* (23) *define r functions $x_i(t)$ which are continuous on the interval $|t - \tau| \leqq l$, have the values ξ_i for $t = \tau$, and satisfy the differential equations* (19).

These solutions can be continued to the right and left of the interval $[\tau - l, \tau + l]$, and the values of t which can be reached by such extensions have a lower bound t_0 and an upper bound t_1 as before, provided the region R is finite and closed.

Through any point $(\tau, \xi_1, \xi_2, \cdots, \xi_r)$ interior to the region R in which the properties of the functions f_i are defined, there passes a system of solutions of the equations (19), *denoted by*

$$(25) \qquad x_i = \phi_i(t, \tau, \xi_1, \xi_2, \cdots, \xi_r) \qquad (i = 1, 2, \cdots, r),$$

or, in the notation of complex numbers,

$$x = \phi(t, \tau, \xi).$$

The functions ϕ_i are continuous in t and define points $(t, x_1, x_2, \cdots, x_n)$ in the interior of R for all values of t in an interval

$$t_0 < t < t_1$$

including the value τ.

The statements made in §1 *concerning the behavior of the solution in the vicinity of* t_0 *and* t_1 *apply also to the solutions* (25).

7. Uniqueness and continuity for a system of equations. The region R_2 (compare §2) in the neighborhood of a system S of solutions $x = x(t)$ which are defined and are interior to R in an interval $[t', t'']$, consists of all the points (t, x) satisfying the inequalities

$$t' \leqq t \leqq t'', \quad \operatorname{mod}[x - x(t)] \leqq \rho.$$

For a properly chosen ρ, R_2 will be entirely interior to R. If $\bar{S}\Big(y = y(t)\Big)$ is another system of solutions interior to R_2 for $t = \tau$, then by §5 b) and (20),

$$\left|\frac{d}{dt}\operatorname{mod}(x - y)\right| \leqq \operatorname{mod}\frac{d(x - y)}{dt} = \operatorname{mod}[f(t, x) - f(t, y)]$$
$$\leqq k_2 \operatorname{mod}(x - y),$$

where k_2 belongs to R_2. As in §2, it follows that

$$\operatorname{mod}(x - y) \leqq e^{k_2|t-\tau|}\cdot\operatorname{mod}(\xi - \eta), \tag{26}$$

where ξ and η are complex numbers representing the values of x and y at $t = \tau$. But mod $(x - y)$ is necessarily $\geqq 0$. Hence *only one system of solutions of equations* (19) *can pass through a given point* $(\tau, \xi_1, \xi_2, \cdots, \xi_n)$ *interior to* R. *The functions* (25) *represent therefore all systems of solutions in the region.*

Let the system $x = x(t)$ be replaced by the system of solutions (25), and let R_2 be so chosen that the interval $[t', t'']$ includes the value τ. If $\Delta\tau$ and a complex number $\Delta\xi = (\Delta\xi_1, \Delta\xi_2, \cdots, \Delta\xi_n)$ are such that the point $(\tau + \Delta t, \xi + \Delta\xi)$ is interior to R_2, then from (26) when τ is replaced by $\tau + \Delta\tau$,

$$\operatorname{mod}[\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)]$$
$$\leqq e^{k_2|t-\tau-\Delta\tau|}\operatorname{mod}[\xi + \Delta\xi - \phi(\tau + \Delta\tau, \tau, \xi)];$$

or by means of (24),

$$\operatorname{mod}[\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)]$$
$$\leqq e^{k_2|t-\tau-\Delta\tau|}\operatorname{mod}\left[\Delta\xi - \int_{\tau}^{\tau+\Delta\tau} f(t, \phi)\,dt\right]$$

But by §4 a) and c), this becomes

$$\operatorname{mod}[\phi(t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)]$$
$$\leqq e^{k_2|t-\tau-\Delta\tau|}[\operatorname{mod}\Delta\xi + M_2|\Delta\tau|], \tag{27}$$

where M_2 is the maximum of mod f in R_2. The inequality (27) holds throughout the interval T including τ, in which the solution $\bar{S}$,

$$x = \phi(t, \tau + \Delta\tau, \xi + \Delta\xi) ,$$

through the point $(\tau + \Delta\tau, \xi + \Delta\xi)$ is defined and interior to R_2.

Choose now $\Delta\tau$ and $\Delta\xi$ so that

$$e^{k_2|t''-t'|}[\operatorname{mod}\Delta\xi + M_2|\Delta\tau|] < \rho. \tag{28}$$

The solution $\bar{S}$ can then be continued over the whole interval $[t', t'']$ because, on account of (28), $\bar{S}$ remains interior to R_2 for all values of t between t' and t''. Furthermore if t and $t + \Delta t$ are both in $[t', t'']$,

$$\begin{aligned} \operatorname{mod}\Delta\phi &= \operatorname{mod}[\phi(t + \Delta t, \tau + \Delta\tau, \xi + \Delta\xi) - \phi(t, \tau, \xi)] \\ &\leqq M_2|\Delta t| + e^{k_2|t-\tau-\Delta\tau|}[\operatorname{mod}\Delta\xi + M_2|\Delta\tau|], \end{aligned}$$

by means of §5 a), (24), and (27), in a manner similar to that of §3. It is readily seen that *the solutions* (25) *are continuous for all values of t and the initial values* $(\tau, \xi_1, \xi_2, \cdots, \xi_r)$ *which define points of the solution interior to the region R.*

8. Systems of linear equations. The system of linear equations

$$\frac{dz_i}{dt} = \sum_{j=1}^{r} a_{ij}z_j \qquad (i = 1, 2, \cdots, r), \tag{29}$$

can be represented in the notation of complex numbers by the single equation

$$\frac{dz}{dt} = Az$$

[see §5, 7), 9), 10)], where z is complex with elements z_i, and A is the matrix of elements $a_{ij}(i, j = 1, 2, \cdots, r)$. The coefficients a_{ij} are supposed to be continuous functions of t on an interval $t_0 \leqq t \leqq t_1$. By the methods ot §§6, 7* *a matrix E can be determined, of which the columns are r systems of solutions of the equations* (29), *taking respectively the columns of the matrix*

$$\begin{matrix} 1, & 0, & 0, & \cdots, & 0 \\ 0, & 1, & 0, & \cdots, & 0 \\ 0, & 0, & 1, & \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & 0, & \cdots, & 1 \end{matrix} \tag{30}$$

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*The region R consists of all points (t, z) for which $t_0 \leqq t \leqq t_1$ and z is arbitrary. The constant k can be taken as r times the maximum of the absolute values of the a^{ij} in the interval $[t_0, t_1]$.

as initial values for $t = \tau$. The elements of E are continuous functions of t and τ when these values lie in the interval $[t_0, t_1]$.

From §5, 12), and the fact that the columns of E are systems of solutions of equations (29), it follows that

$$\frac{dE}{dt} = AE. \tag{31}$$

Consider the set of functions

$$z = E\zeta,$$

where ζ is a complex constant. By differentiation and application of (31) [see §5, 7), 13), 10), 12)],

$$\frac{dz}{dt} = \frac{dE}{dt}\zeta = AE\zeta = Az.$$

Hence *the system of integrals of equations* (29) *taking any given initial values* ζ *for $t = \tau$, and which by* §7 *is unique, can be expressed as the product of the matrix E and the initial values ζ,*

$$z = E\zeta.$$

In the next section it will be important to consider the solutions of linear equations in which the coefficients a_{ij} involve a number of parameters besides t. Use will be made of the following

Auxiliary Theorem: *If the coefficients a_{ij} of the equations* (29) *involve also s parameters $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s)$, and are continuous functions of them in a finite closed region* P, *then the elements of the matrix E are continuous functions of t, τ, α for all values of t, τ, α such that t and τ are in the interval* $[t_0, t_1]$ *and α in* P.

The proof is easily made by examining the set of approximation functions of the form (3). Each of them is defined on the whole interval $[t_0, t_1]$ (see footnote p. 60), and is a continuous function of t, τ, α. If the constant N of §6 is taken as the maximum of mod $f(t, \zeta) =$ mod $A\zeta$ for all values of t in $[t_0, t_1]$ and of α in P, then the convergence proof is independent of the parameters α, and the series corresponding to (23) are uniformly convergent series of continuous functions of t, τ, α; which proves the theorem.*

9. Partial derivatives of a system of solutions. The existence of the first partial derivatives of the solutions (25) with respect to the initial

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*See Jordan, *loc. cit.* p. 311.

constants can be proved, provided that the functions $f_i(t, x_1, x_2, \cdots, x_n)$ have continuous first partial derivatives with respect to the x's in the interior of the region R. Let $\bar{t}, \bar{\tau}, \bar{\xi}$ define a point of (25) interior to R, and suppose R_2 constructed as in §7 so that the interval $[t', t'']$ includes the values $\bar{t}$ and $\bar{\tau}$. The difference of the two solutions $\phi, \phi + \Delta\phi$ corresponding to $(\bar{\tau}, \bar{\xi})$ and $(\bar{\tau} + \Delta\tau, \bar{\xi} + \Delta\xi)$ respectively, satisfies the equation

$$(32) \qquad \frac{d\Delta\phi}{dt} = f(t, \phi + \Delta\phi) - f(t, \phi) = A\Delta\phi,$$

where the elements of the matrix A are determined by the equations

$$\begin{aligned}\Delta\phi_j \cdot a_{ij} = f_i(t, \phi_1, \cdots, \phi_{j-1}, \phi_j + \Delta\phi_j, \cdots, \phi_r + \Delta\phi_r)\\ - f_i(t, \phi_1, \cdots, \phi_{j-1}, \phi_j, \phi_{j+1} + \Delta\phi_{j+1}, \cdots, \phi_r + \Delta\phi_r).\end{aligned}$$

If $\bar{\tau}$ and $\bar{\xi}$ are considered fixed for the moment, it can be shown (compare with §4) that the coefficients a_{ij} are continuous functions of $t, \Delta\tau, \Delta\xi$, provided that

$$(33) \qquad t' \leqq t \leqq t'', \quad |\Delta\tau| = |\tau - \bar{\tau}| \leqq \delta, \quad \operatorname{mod}\Delta\xi = \operatorname{mod}(\xi - \bar{\xi}) \leqq \delta,$$

where δ is so chosen that the point $(\bar{\tau} + \Delta\tau, \bar{\xi} + \Delta\xi)$ is in R_2 and the inequality (28) is satisfied. This follows from the fact that the solutions $\phi + \Delta\phi$ corresponding to such values of $\Delta\tau$, $\Delta\xi$ are defined and interior to R_2 in the interval $[t', t'']$, and according to §7 the functions ϕ are continuous.

By the auxiliary theorem of §8 a matrix E exists for the equations (32), with elements which take the initial values (30) at $t = \bar{\tau}$, and which are continuous functions of $t, \Delta\tau, \Delta\xi$ in the region (33). Suppose that $\Delta\xi_1$ is different from zero, while $\Delta\tau = \Delta\xi_2 = \Delta\xi_3 = \cdots = \Delta\xi_r = 0$. The r quotients $\dfrac{\Delta\phi}{\Delta\xi_1}$ [see §5, 6)] are a system of integrals of equations (32) with the initial values $(1, 0, 0, \cdots, 0)$ for $t = \bar{\tau}$, and therefore coincide with the first column of E. A similar reasoning applies to the other systems $\dfrac{\Delta\phi}{\Delta\xi_j}$ $(j = 2, \cdots, r)$ for which $\Delta\tau$ and all the elements of $\Delta\xi$ except $\Delta\xi_j$ are set equal to zero. Since the elements of E are continuous functions, it follows for each system that the limits of the quotients $\dfrac{\Delta\phi_i}{\Delta\xi_j}$ as $\Delta\xi_j$ approaches zero exist, and at the limit

$$\frac{\partial\phi}{\partial\xi} = E\Big|^{\Delta\tau=\Delta\xi=0}, \tag{34}$$

where $\frac{\partial\phi}{\partial\xi}$ is the matrix of derivatives $\frac{\partial\phi_i}{\partial\xi_j}$ $(i, j = 1, 2, \cdots, r)$.

So far only the existence of the derivatives (34) has been proved, but it can be readily shown that they are also continuous at the given point $\bar{t}$, $\bar{\tau}$, $\bar{\xi}$. By substituting $x = \phi$ in the equations (19) and deriving successively for ξ_i $(i = 1, 2, \cdots, r)$, it follows that the columns of the matrix $\frac{\partial\phi}{\partial\xi}$ are solutions of the equations

$$\frac{dz}{dt} = f_x z, \tag{35}$$

where f_x denotes the matrix of elements $\frac{\partial f_i}{\partial x_j}$ $(i, j = 1, 2, \cdots, r)$ in which the arguments are t, $\phi(t, \bar{\tau}, \bar{\xi})$. Equations (32) go over into these equations when $\Delta\tau = \Delta\xi = 0$. If the $\bar{\tau}$ and $\bar{\xi}$ are replaced by $\tau = \bar{\tau} + \Delta\tau$ and $\xi = \bar{\xi} + \Delta\xi$, the coefficients $\frac{\partial f_i}{\partial x_j}$ will be continuous functions of t, τ, ξ in the region (33). The elements of the matrix of solutions $\frac{\partial\phi}{\partial\xi}$ with the initial values (30) at $t = \tau$, are therefore continuous functions of t, τ, ξ in the region (33), in particular at the point $(\bar{t}, \bar{\tau}, \bar{\xi})$.

When $\Delta\tau \neq 0$, $\Delta\xi = 0$, the value of $\Delta\phi$ at $t = \tau$ is found from equation (24) to be

$$\Delta\phi\Big|^{t=\bar{\tau}} = \int_{\tau+\Delta\tau}^{\bar{\tau}} f(t, \phi + \Delta\phi)\,dt = -\,\Delta\tau \cdot \bar{f}, \tag{36}$$

where the elements of the complex number $\bar{f}$ are

$$f_i\Big(\bar{\tau} + \theta_i\Delta\tau,\ \phi(\bar{\tau} + \theta_i\Delta\tau,\ \bar{\tau} + \Delta\tau,\ \xi)\Big),\quad 0 < \theta_i < 1,\ (i = 1, 2, \cdots, r).$$

Accordingly the quotients $\frac{\Delta\phi}{\Delta\tau}$, which form a system of solutions of equations (32) with initial values at $t = \tau$ defined by (36), have the values (see §8)

$$\frac{\Delta\phi}{\Delta\tau} = -\,E\bar{f}.$$

At the limit for $\Delta\tau = 0$,

$$\frac{\partial\phi}{\partial\tau} = -\,\frac{\partial\phi}{\partial\xi}\,f(\tau, \xi), \tag{37}$$

where the right member is the product of the matrix $\frac{\partial\phi}{\partial\xi}$ and the complex number $f(\tau, \xi)$ [see §5, 6) and 10)].

It has been proved therefore that *when the functions $f_i(t, x_1, x_2, \cdots, x_n)$ have continuous first partial derivatives with respect to the x_i's in the interior of the region R, then the solutions* (25) *have first partial derivatives which are continuous for all values $\bar{t}, \bar{\tau}, \bar{\xi}$ defining points of the solution interior to R.*

The existence of higher derivatives at a point $\bar{t}, \bar{\tau}, \bar{\xi}$ when further assumptions are made upon the functions f_i, can be proved by means of the $2r + 1$ equations,

$$\frac{d\tau}{dt} = 0, \quad \frac{d\xi}{dt} = 0, \quad \frac{dz}{dt} = f_x z, \tag{38}$$

where the arguments of f_x are $t, \phi(t\ \tau, \xi)$. The right members are functions of t, τ, ξ, z, and are continuous when t, τ, ξ satisfy the conditions (33) whatever the value of the arguments z. If the f_i have continuous second partial derivatives with respect to the x_i's, then according to the last theorem the right members of (38) have continuous first partial derivatives for τ, ξ, z, and their solutions

$$\tau = \tau, \quad \xi = \xi, \quad z = \frac{\partial\phi}{\partial\xi_i}$$

are therefore differentiable for the initial values τ, ξ. If furthermore the f_i have continuous first partial derivatives with respect to t, then on account of the differentiability of the functions $\frac{\partial\phi_i}{\partial\xi_j}$ and f_i, the expressions (37) can be differentiated once, and ϕ twice, for τ. From (24) the functions φ can also be differentiated twice for t. In general the following theorem can be proved:

If the functions $f_i(t, x_1, x_2, \cdots, x_r)$ have all the partial derivatives of order $n - 1$, and all of order n except perhaps the n^{th} derivatives with respect to t alone, continuous in the region R, then the solutions (25) *of equations* (19) *have continuous partial derivatives of the n^{th} order for all values of t, τ, ξ defining points of the solution* (25) *interior to R.*

For suppose that it holds for n, and that the functions f_i have all derivatives of the n^{th} order and all of order $n + 1$ except perhaps the $(n + 1)^{st}$ derivatives with respect to t. Then according to the hypothesis the functions ϕ have continuous n^{th} derivatives, and, on account of the existence of the $(n + 1)^{st}$ derivatives of the functions f_i, the right members of equations (38) can be differentiated n times for t, τ, ξ, and z. The solutions $\frac{\partial \phi}{\partial \xi}$ and the expressions (37) can therefore also be differentiated n times. Since the functions f_i have continuous n^{th} derivatives with respect to t alone, equation (24) shows that the functions ϕ_i can be differentiated $n + 1$ times for t alone. The induction is therefore complete, and the theorem holds for any value of n.

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